

# Tachyonic Dirac sea

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## Abstract

We consider a system of many fermions with tachyonic energy spectrum  $\varepsilon_k = \sqrt{k^2 - m^2}$  and clarify that tachyons with imaginary energy and low momentum ( $k < m$ ) play the role of Dirac sea in a many-tachyon Fermi system and make contribution to the thermodynamical functions. The energy and pressure acquire additional constant terms that, however, is not reflected in the sound speed. Replacement  $m \mapsto im$  results in the thermodynamical functions and the sound speed of an ordinary Fermi gas. When the Fermi momentum approaches the Dirac sea level  $k_F \rightarrow m$ , the group velocity of most tachyons above the sea is unbound, while the sound speed tends to infinity. This scenario is not encountered in practice because the cold tachyon Fermi gas becomes unstable with respect to hydrodynamical perturbations as soon as  $k_F < \sqrt{3/2}m$ . The particle number density of a stable many-tachyon system is always finite and exceeds the critical value depending on the tachyon mass  $m$ .

## 1 Introduction

Tachyons are instabilities of the field theory often considered in cosmological models. The tachyonic Dirac equation [1, 2, 3, 4, 5]

$$(i\gamma^\mu \partial_\mu - \gamma_5 m) \psi = 0 \quad (1)$$

has plane-wave solution

$$\psi \sim \exp\left(i\vec{k} \cdot \vec{r} - i\varepsilon_k t\right) \quad (2)$$

with the energy spectrum

$$\varepsilon_k = \sqrt{k^2 - m^2} \quad (3)$$

that can be presented in the form

$$\varepsilon_k = \text{Re}\varepsilon_k + i\text{Im}\varepsilon_k = \sqrt{k^2 - m^2}\Theta(k - m) + i\sqrt{m^2 - k^2}\Theta(m - k) \quad (4)$$

where

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (5)$$

is the Heaviside step function. It is clear that solution (2) at  $\text{Im}\varepsilon_k = 0$  describes a stationary plane wave  $\|\psi\| = \text{const}$ , while

$$\|\psi\| = \sqrt{\psi^\dagger \psi} \sim \exp(-\text{Im}\varepsilon_k t) \quad (6)$$

corresponds to an unstable particle at  $\text{Im}\varepsilon_k \neq 0$  ( $k < m$ ).

A system of many tachyons can be studied in the frames of statistical mechanics [6, 7, 8, 9]. A single tachyon with imaginary energy (4) and small momentum ( $k < m$ ) cannot be presented by a stable plane wave (2) because the amplitude of its wave function is subject to decay (6). How to operate with unstable sector  $k < m$  when a many-particle system is considered? Indeed, we can exclude unstable particles from consideration at all and operate with the only stable particles whose momentum  $k \geq m$ . However, a Fermi gas of many tachyons may contain occupied states at low momentum  $k < m$  if they satisfy the Pauli exclusion principle. When we estimate the thermodynamical functions of a cold tachyon Fermi gas, should we take into account or disqualify the "unphysical" states with imaginary energy ( $k < m$ )? This problem is clarified in the present paper.

## 2 Thermodynamical functions

Starting with the thermodynamical potential [10]

$$\Omega = -\frac{\gamma T}{2\pi^2} V \int_0^\infty \ln \left( 1 + \exp \frac{\mu - \varepsilon_k}{T} \right) k^2 dk \quad (7)$$

for a system of constant number of  $N$  fermions we determine the Helmholtz free energy  $F = \Omega + \mu N$ , the pressure

$$\begin{aligned} P &= -\frac{\Omega}{V} = \frac{\gamma}{2\pi^2} T \int_0^\infty \ln \left( 1 + \exp \frac{\mu - \varepsilon_k}{T} \right) k^2 dk = \\ &= \frac{\gamma T}{6\pi^2} k^3 \ln \left( 1 + \exp \frac{\mu - \varepsilon_k}{T} \right) \Big|_0^\infty + \frac{\gamma}{6\pi^2} \int_0^\infty f_\varepsilon \frac{d\varepsilon_k}{dk} k^3 dk \end{aligned} \quad (8)$$

the energy density

$$E = -\frac{T^2}{V} \frac{\partial (F/T)_{V,\mu}}{\partial T} = \frac{\gamma}{2\pi^2} \int_0^\infty f_\varepsilon \varepsilon_k k^2 dk \quad (9)$$

and the particle number density

$$n = \frac{N}{V} = -\frac{1}{V} \left( \frac{\partial \Omega}{\partial \mu} \right)_{V,T} = \frac{1}{V} \frac{\partial (T \ln Z)_{V,T}}{\partial \mu} = \frac{\gamma}{2\pi^2} \int_0^\infty f_\varepsilon k^2 dk \quad (10)$$

where the Fermi-Dirac distribution function is

$$f_\varepsilon = \frac{1}{\exp [(\varepsilon_k - \mu) / T] + 1} \quad (11)$$

and the chemical potential  $\mu$  satisfies relation

$$\mu = \left( \frac{\partial F}{\partial N} \right)_{T,V} \quad (12)$$

Formulas (8)-(10) at zero temperature yield the third law of thermodynamics (Nernst heat theorem)

$$E + P = \mu n \quad (13)$$

while formula (12) is reduced to  $\mu = dE/dn$ , and the sound speed at zero temperature is determined by formula

$$c_s^2 = \frac{dP}{dE} = \frac{n}{\mu} \frac{d\mu}{dn} \quad (14)$$

Formula (8) is reduced to

$$P = \frac{\gamma}{6\pi^2} \int_0^\infty f_\varepsilon \frac{d\varepsilon_k}{dk} k^3 dk \quad (15)$$

under assumption

$$\lim_{k \rightarrow \infty} \varepsilon_k = +\infty \quad \lim_{k \rightarrow 0} k^3 \varepsilon_k = 0 \quad (16)$$

Most particles and quasi-particles satisfy condition (16), particularly, it is so for the tachyonic energy spectrum (3).

### 3 Cold tachyon Fermi gas

We may expect that the chemical potential is a complex quantity  $\mu = \text{Re}\mu + i\text{Im}\mu$ . If we expect that  $\text{Re}\mu > 0$  and  $\text{Im}\mu = 0$ , then, the distribution function (11) at zero temperature is reduced to

$$f_\varepsilon = \Theta(\varepsilon_F - \varepsilon_k) \quad (17)$$

that is equivalent to

$$f_\varepsilon = \Theta(k_F - k) \quad (18)$$

where

$$\varepsilon_F = \text{Re}\mu|_{T=0} \quad (19)$$

is the Fermi energy corresponding to the Fermi momentum  $k_F$  according to relation

$$\varepsilon_F = \sqrt{k_F^2 - m^2} \quad (20)$$

Substituting the distribution function at zero temperature (18) in (10), we have

$$n = \frac{\gamma}{2\pi^2} \int_0^{k_F} k^2 dk = \frac{\gamma k_F^3}{6\pi^2} = \frac{\gamma \sqrt{(\varepsilon_F^2 + m^2)^3}}{6\pi^2} \quad (21)$$

that at  $\varepsilon_F \rightarrow 0$  tends to

$$n \rightarrow n_\star = \frac{\gamma m^3}{6\pi^2} \quad (22)$$

According to (14) and (21) we find the sound speed

$$c_s^2 = \frac{1}{3} \frac{k_F^2}{k_F^2 - m^2} \quad (23)$$

that is superluminal when  $k_F < \sqrt{3/2}m$  corresponds to

$$n < n_T = \sqrt{\frac{27}{8}} n_\star \simeq 1.84 n_\star \quad (24)$$

Substituting the tachyonic energy spectrum (4) and the distribution function (18) in (9) and in (15), we find the energy density and pressure at zero temperature

$$E = \frac{\gamma}{2\pi^2} \int_m^{k_F} k^2 \text{Re}\varepsilon_k dk + \frac{i\gamma}{2\pi^2} \int_0^m k^2 \text{Im}\varepsilon_k dk \quad (25)$$

$$P = \frac{\gamma}{6\pi^2} \int_m^{k_F} k^3 \frac{d\text{Re}\varepsilon_k}{dk} dk + \frac{i\gamma}{6\pi^2} \int_0^m k^3 \frac{d\text{Im}\varepsilon_k}{dk} dk \quad (26)$$

Imaginary constants

$$E_0 = \frac{i\gamma}{2\pi^2} \int_0^m k^2 \text{Im}\varepsilon_k dk = \frac{i\gamma}{2\pi^2} \int_0^m k^2 \sqrt{m^2 - k^2} dk = \frac{i\gamma m^4}{32\pi} \quad (27)$$

$$P_0 = \frac{i\gamma}{6\pi^2} \int_0^m k^3 \frac{d\text{Im}\varepsilon_k}{dk} dk = -\frac{i\gamma}{2\pi^2} \int_0^m \frac{k^4}{\sqrt{m^2 - k^2}} dk = -\frac{i\gamma m^4}{32\pi} \quad (28)$$

are included in the energy density (25) and pressure (26), playing the role of zero-point levels. Notable that the multiplier  $5/(32\pi)$  appears in the imaginary part of the vacuum energy of tachyonic modes in the Chern-Simons electrodynamics [11]

We can exclude the 'vacuum' imaginary terms, making renormalization

$$\bar{E} \mapsto E - E_0 \quad (29)$$

$$\bar{P} \mapsto P - P_0 \quad (30)$$

The real parts of energy density (25) and pressure (26) were determined in [9]:

$$\begin{aligned}\bar{E} &= \frac{\gamma}{2\pi^2} \int_m^{k_F} k^2 \text{Re}\varepsilon_k dk = \frac{\gamma}{2\pi^2} \int_m^k k^2 \sqrt{k^2 - m^2} dk = \\ &= \frac{\gamma}{8\pi^2} k_F^3 \varepsilon_F - \frac{\gamma}{16\pi^2} m^2 \left( k_F \varepsilon_F + m^2 \ln \frac{k_F + \varepsilon_F}{m} \right)\end{aligned}\quad (31)$$

$$\begin{aligned}\bar{P} &= \frac{\gamma}{6\pi^2} \int_m^{k_F} k^3 \frac{d\text{Re}\varepsilon_k}{dk} dk = \frac{\gamma}{6\pi^2} \int_m^{k_F} \frac{k^4}{\sqrt{k^2 - m^2}} dk = \\ &= \frac{\gamma}{24\pi^2} k_F^3 \varepsilon_F + \frac{\gamma}{16\pi^2} m^2 \left( k_F \varepsilon_F + m^2 \ln \frac{k_F + \varepsilon_F}{m} \right)\end{aligned}\quad (32)$$

and it is also enough for calculation of the sound speed  $c_s^2 = dP/dE$  that coincides with (23) because constants  $E_0$  and  $P_0$  in (29)- (30) do not influence the result of differentiation.

## 4 Alternative view

One should not hesitate to consider the tachyonic energy spectrum (3) in the whole range of momentum  $k$  where the energy includes imaginary part (4), in fact, tachyonic states with complex energy are known in the solid state physics [12]. The fact that we determine the right energy density (25) and pressure (26) is immediately checked by formal replacement  $m \mapsto im$  that yields the thermodynamical functions of an ordinary relativistic Fermi gas

$$E = \frac{\gamma}{2\pi^2} \int_0^{k_F} k^2 \sqrt{k^2 + m^2} dk = \frac{\gamma}{8\pi^2} k_F^3 \varepsilon_F + \frac{\gamma}{16\pi^2} m^2 \left( k_F \varepsilon_F - m^2 \ln \frac{k_F + \varepsilon_F}{m} \right)\quad (33)$$

$$P = \frac{\gamma}{2\pi^2} \int_0^{k_F} k^2 \sqrt{k^2 + m^2} dk = \frac{\gamma}{24\pi^2} k_F^3 \varepsilon_F - \frac{\gamma}{16\pi^2} m^2 \left( k_F \varepsilon_F - m^2 \ln \frac{k_F + \varepsilon_F}{m} \right)\quad (34)$$

constituted of subluminal particles with the energy spectrum  $\varepsilon_k = \sqrt{k^2 + m^2}$ . The same conversion is applied to the sound speed (23).

It should be also noted that the tachyonic sound speed (23) becomes infinite when  $k_F \rightarrow m$  that corresponds to infinite group velocity

$$v = \frac{d\varepsilon_k}{dk} = \frac{k}{\sqrt{k^2 - m^2}} \rightarrow \infty \quad (35)$$

for all stable tachyons with real energy ( $k > m$ ).

If the integration is truncated in the very beginning within the range  $k \in (m, \infty)$ , then, the energy density (9) remains the same (31), while the pressure (15) will be

$$\begin{aligned} P &= \frac{\gamma}{2\pi^2} T \int_m^{k_F} k^2 \ln \left( 1 + \exp \frac{\varepsilon_F - \varepsilon_k}{T} \right) dk = \\ &= \frac{\gamma}{24\pi^2} k_F^3 \varepsilon_F + \frac{\gamma}{16\pi^2} m^2 \left( k_F \varepsilon_F + m^2 \ln \frac{k_F + \varepsilon_F}{m} \right) - \frac{\gamma m^3}{6\pi^2} \varepsilon_F \end{aligned} \quad (36)$$

Now conversion to the ordinary Fermi gas (33)-(34) by replacement  $m \mapsto im$  is not working. The relevant sound speed [13, 14]

$$c_s^2 = \frac{dP}{dE} = \frac{1}{3} \frac{k_F^2 + mk_F + m^2}{k_F^2 + mk_F} \quad (37)$$

is finite at any  $k_F$  that seems unrealistic because all tachyons of the thermodynamical ensemble are traveling at infinite high velocity (35) when  $k_F \rightarrow m$ . All this implies that unstable sector  $k < m$  should not be omitted when we analyze the properties of a many-tachyon Fermi system.

## 5 Conclusion

Although a single tachyon has real energy (3) at large momentum  $k > m$ , a system of tachyons in a thermodynamical ensemble obeys the Fermi-Dirac statistics and unstable states with imaginary energy ( $k < m$ ) should be included in analysis.

The properties of a cold tachyon Fermi gas depend on its density. The medium is stable with respect to hydrodynamical perturbations only at high density  $n \geq n_T$  (31). When we consider a stable system of many fermions

with tachyonic energy spectrum (3), the quantum states  $k < m$  are responsible for imaginary constants (27) and (28) added to the thermodynamical functions (25) and (26). It implies reset of the zero-point levels of the energy density (29) and pressure (30), however, the sound speed (23) is independent on them.

The tachyons with imaginary energy and small momentum  $k < m$  are inherent in any many-tachyon Fermi system and we can imagine them in the tachyonic Dirac sea (see Fig. 1) because all tachyons from the whole sea ( $k < m$ ) are making contribution to the thermodynamical functions even when the Fermi momentum itself is above the sea level  $k_F > m$ . Operating with the only real part of tachyonic energy spectrum (4), we should be careful to complete this mathematical trick for determining the real part of the energy density (31) and pressure (32), and, hence, again coming to the sound speed (23) [9]. However, if we exclude the "unphysical" particles from the sea ( $k < m$ ), we come to the tachyonic thermodynamical functions (31) and (36) that are not convertible to the thermodynamical functions of an ordinary Fermi gas (33)-(34) after replacement  $m \mapsto im$ , while the relevant sound speed (37) remains finite even at  $k_F \rightarrow m$  when most "physical" tachyons ( $k > m$ ) travel at infinite high velocity (37).

Now we have seen the validity of formulas for the energy density (25), pressure (26), sound speed (23). It is important that the causality condition  $c_s^2 \leq 1$  is satisfied at high density  $n > n_T$  (24) which is definitely higher than  $n_*$  (22) corresponding to  $k_F = m$ . The properties of a cold tachyon Fermi gas at small density  $n < n_*$  may attract only pure theoretical interest because the system becomes unstable as soon as its density decreases below  $n = n_T \simeq 1.84n_*$ , and applied problems, like the tachyon stars [15], require the only knowledge of stable tachyon Fermi gas.

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Figure 1: Graphical illustration of the Dirac sea of massive subluminal particles (a), massless particles (b) and tachyons (c).

